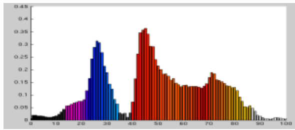


Signals and Systems
Lecture 13: Fourier Series Representation of Periodic Signals (Part 1)

Outline

- **Introduction.**
- **Defining Equations for Fourier series.**
  - ✓ Trigonometric Fourier series.
  - ✓ Compact Trigonometric Fourier series (FS).
  - ✓ Exponential Fourier series.
- **Examples**

Introduction

- **Baron Jean Baptiste Joseph Fourier (1768−1830) introduced the idea that any periodic function can be represented by a series of sines and cosines, which are harmonically related.**

Baron Jean Baptiste Joseph Fourier (1768−1830)

- **A continuous time signal $x(t)$ is said to be periodic if $x(t) = x(t+T)$, for all t.**
  - **Examples of basic periodic signals:**

$$x(t) = cos\,(\omega_0 t) \rightarrow \text{Real sinusoid.}$$

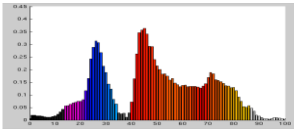
$$x(t) = e^{j\omega_0 t} \rightarrow \text{Complex exponential}$$

Where

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

  - $e^{jk\omega_0 t} \Rightarrow$ **Harmonically Related** $\Rightarrow \frac{T_0}{k} = \frac{2\pi}{k\omega_0}$

- **An approximation of the periodic signal $x(t)$ can built up by adding appropriate combination of harmonics to the fundamental harmonic, this sum is called a Fourier series.**



Defining Equations for Fourier series

1) Trigonometric Fourier series (Trigonometric CTFS)

- Arbitrary periodic signal can be expressed as a linear combination of sinusoidal signals, the periodic signal $x(t)$ can be split up as Sines and Cosines of fundamental frequency ω_0 and all of its harmonies and expressed as given below:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \, .cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \, .sin(k\omega_0 t)$$

- ✓ Where k from 1 to ∞ $\Rightarrow$ **Positive frequency.**
- ✓ a_0- Zero harmonic or Dc, Average value.
- ✓ Where $\omega_0 = \frac{2\pi}{T_0}$ is the fundamental frequency of $x(t)$ and coefficients a_0, a_k, and b_k are referred to as the **trigonometric CTFS coefficients.** The coefficients are calculated as follows:

$$a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) dt$$

$$a_k = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t). \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T_0} \int_{\langle T_0 \rangle} x(t). sin(k\omega_0 t) \, dt$$

$$T_0 = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$$

T_0- **Fundamental period in seconds**
f_0- **Fundamental frequency in Hertz**
ω_0- **Radian frequency in rad/sec**

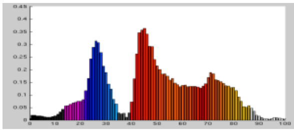
2) Compact Trigonometric Fourier series (FS)

- The **trigonometric FS** can be represented in compact form. It is also called polar FS:
- "When $x(t)$ is **real**, the coefficients of trigonometric form a_k and b_k are **real**.
- The compact form of FS is given by the following

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \, .cos(k\omega_0 t - \theta_k)$$

- ✓ Where k from 1 to ∞ $\Rightarrow$ **Positive frequency.**
- ✓ **Where:**

$$C_0 = a_0 = \frac{1}{T_0} \int_{\langle T_0 \rangle} x(t) dt$$



$$C_k = \sqrt{a_k^2 \;\; + b_k^2}$$

$$\theta_k = tan^{-1}\left(\frac{b_k}{a_k}\right)$$

3) Exponential Fourier series

- General periodic signal $x(t)$ can be represented as a **linear combination** of harmonically **related complex exponentials**

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \,.\, e^{jk\omega_0 t}$$

  - ✓ a_k – **Complex:**

$$a_k = A_k e^{j\theta_k} = B_k + jC_k$$
$$a_k = A_k e^{j\theta_k} \Rightarrow \textit{Polar form}$$
$$a_k = B_k + jC_k \Rightarrow \textit{Rectangular form}$$

  - ✓ Where k from $-\infty$ to ∞ $\Rightarrow$ **positive and negative frequency.**
  - ✓ $k\omega_0$ - Frequencies that are **positive and negative.**
- Two Questions are **important** in this form:
  1) How to **determine** these coefficients.
  2) How a **broad of class signal**, can be represented using exponential FS.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \,.\, e^{jk\omega_0 t} \quad \Rightarrow \quad (\textit{Synthesis equation})$$

$$a_k = \frac{1}{T_0}\int_{\langle T_0\rangle} x(t).\, e^{-jk\omega_0 t} \quad \Rightarrow \quad (\textit{Analysis equation})$$

- **Note:** Using Euler's identity, the trigonometric form can be represented using exponential form:

$$e^{jk\omega_0 t} = cos\,(k\omega_0 t) \; + \; j\,sin\,(k\omega_0 t)$$

- **Note:** a_k calculation in Exponential form is much easier compared to a_0, a_k , b_k in trigonometric form.
- $x(t)$ and a_k are represented by the **FS pair**, as

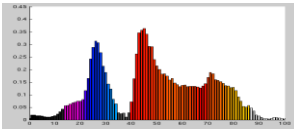
$$x(t) \;\Leftrightarrow\; a_k.$$
$$x(t) \;\Leftrightarrow\; X(k) \qquad \text{different sources/Books.}$$

- $X(k)$ are related to **trigonometric FS coefficients** a_0, a_k ,and $b_k$as:

$$X(0) = a_0$$
$$X(k) = \frac{1}{2}(a_k - jb_k)$$
$$X^*(k) = \frac{1}{2}(a_k + jb_k)$$

Where $X^*(k)$ **conjugate of** $X(k)$.

- **Notes related to trigonometric FS form 1**
  - ➢ If the periodical function $x(t)$ is symmetrical with **respect to the time axis**, the coefficient $a_0 = 0$.



- If $x(t)$ is **even function**, only **cosine terms** in FS exist and therefore $b_k = 0$.
- If $x(t)$ is **odd function**, only **sine terms** in FS exist, then $a_k = 0$.

- In the sciences and engineering, the process of **decomposing** a function (signal) into simpler pieces is often called **Fourier analysis**, while the operation of **rebuilding** the function from these pieces is **Fourier synthesis**.
- Synthesis equation: $x(t) = \sum_{k=-\infty}^{\infty} a_k \, . e^{jk\omega_0 t}$ can be rewritten as:

$$x(t) = a_0 + (a_{-1}\, e^{-j\omega_0 t} + a_1 e^{j\omega_0 t}) + a_{-2}\, e^{-j2\omega_0 t} + a_2 e^{j2\omega_0 t}) + \cdots$$

  - Where a_0 → the first component → dc value.
  - The second component $(a_{-1}\, e^{-j\omega_0 t} + a_1 e^{j\omega_0 t})$ is referred to as the **first harmonic** with ω_0.
  - The third component is **second harmonic** with twice fundamental frequency $2\omega_0$.
  - In general the component for $k = \pm N$ are referred to as the **N-th harmonic** with Frequency $N\omega_0$.

Examples

1) **Find the Fourier series coefficients for the signal $x(t) = cos\,(\omega_0 t)$.**

Solution: Using Euler's relation:

$$x(t) = cos\,(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

The fundamental frequency is ω_0 and

$$a_0 = 0 \text{ (dc value)}$$
$$a_1 = \frac{1}{2}$$
$$a_{-1} = \frac{1}{2}$$
$$a_k = 0 \quad for \quad k \neq \pm 1$$

Note that

$$a_{-k} = a_k^*$$

2) **Find the Fourier Series coefficients for $x(t) = sin\,(\omega_0 t)$.**

Solution:

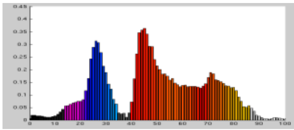
$$x(t) = sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

The fundamental frequency is ω_0 and

$$a_0 = 0 \text{ (dc value)}$$
$$a_1 = \frac{1}{2j}$$
$$a_{-1} = \frac{-1}{2j}$$
$$a_k = 0 \quad for \quad k \neq \pm 1$$

Note that

$$a_{-k} = a_k^*$$



3) Find the Fourier series coefficients for

$$x(t) = 1 + \frac{1}{6}\cos(2\pi t) + \frac{1}{3}\cos(4\pi t) + \cos(6\pi t)$$

Solution:

$$x(t) = 1 + \frac{1}{6}\cos(2\pi t) + \frac{1}{3}\cos(4\pi t) + \cos(6\pi t) =$$

$$1 + \frac{1}{12}e^{j2\pi t} + \frac{1}{12}e^{-j2\pi t} + \frac{1}{6}e^{j4\pi t} + \frac{1}{6}e^{-j4\pi t} + \frac{1}{2}e^{j6\pi t} + \frac{1}{2}e^{-j6\pi t}$$

The fundamental frequency is $\omega_0 = 2\pi$ **and**

$a_0 = 1$ *(dc value)*

$$a_1 = a_{-1} = \frac{1}{12}$$

$$a_2 = a_{-2} = \frac{1}{6}$$

$$a_3 = a_{-3} = \frac{1}{2}$$

$$a_k = 0 \quad for \quad k \neq 0, \pm 1, \pm 2, \pm 3$$

$$x(t) = \sum_{k=-3}^{3} a_k . e^{jk2\pi}$$